What was Apollonius Thinking?

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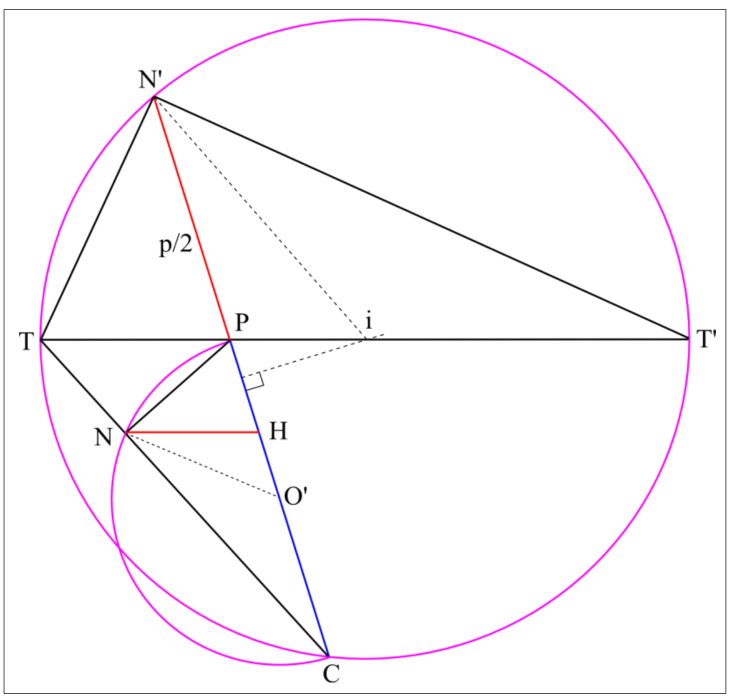
In Heath's version of Apollonius' Conics the preliminaries to proposition I.58 includes this: "…let the given angle not be a right angle but equal to the angle CPT, where C is the middle point of the given diameter PP'; and let PL be the parameter corresponding to PP'. Take a point N, on the semicircle which has CP for its diameter, such that NH drawn parallel to PT satisfies the relation sq. NH : rect. CH,HP :: PL : PP'."

Heath indicates the proof proceeds without elaboration but also notes that Euoticus in the 5th or 6th century wrote a commentary on the Conics which includes a method for constructing NH. The clearest description of Euoticus' method is found in Colin McKinney's 2010 doctoral dissertation "*Conjugate diameters: Apollonius of Perga and Eutocius of Ascalon*" available from the University of Iowa at http://ir.uiowa.edu/etd/711/.

Relative to the construction of NH Heath references Eutocius method to wit:

*'This construction is assumed by Apollonius without any explanation; but we may infer that it was arrived at by a method similar to that adopted for a similar case in Prop. 62. In fact the solution given by Eutocius represents sufficiently closely Apollonius' **probable** procedure."*

Obviously I do not know what Apollonius was thinking. There are, however, methods for constructing NH that are much simpler than that given by Eutocius. I prove one here and leave it to the reader to conclude what Apollonius' probable method may have been.



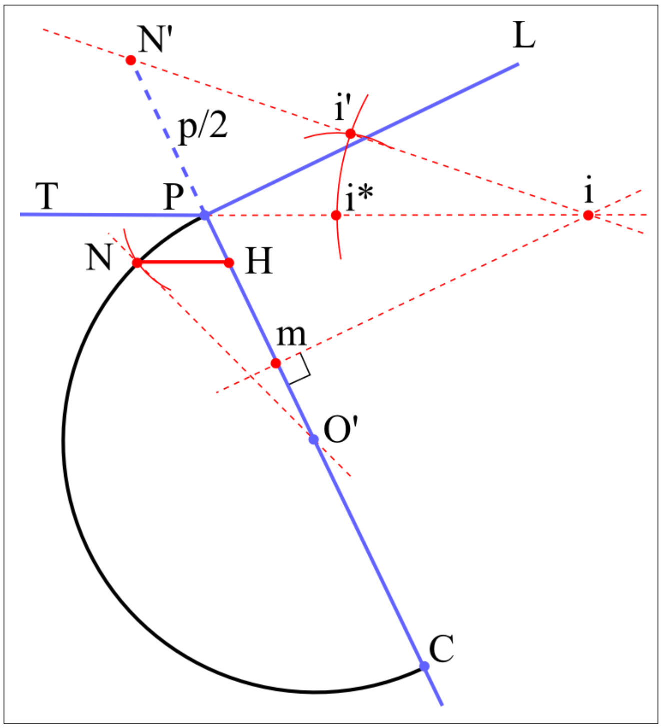
In the figure to the left we began as Apollonius by laying out the ordinate angle CPT with PT of arbitray length. Extend CP by a length of p/2 to the point N'.

At the midpoint of CN' erect a perpendicular line and extend it to intersect the extension of TP at i.

With i as center, construct a circle of radius iN' = iC cutting PT at T and thus fixing the point T. Extend Ti to meet the circle at T'.
Construct a semi circle on CP. Draw lines T'N', N'T, and TC. Line TC will intersect the semi circle at N. Draw NP. ∠CNP will be right. Finally draw NH parallel to TP. NH is the required line.

Proof: ∠TiN' and ∠TCN' both subtend the arc TN' hence = ∠TiN' = 2·∠TCN'. Similarly, ∠NO'P and ∠NCP both subtend the arc NP and thus NO'P = 2·∠NCP = 2·∠TCN' = ∠TiN'.
△TiN' and △NO'P are isosceles with equal apex angles which makes their base angles equal and △TN'i ∽△PNO' which in turn makes ∠NPC = ∠N'TT'. Both ∠TN'T' and ∠PNC are right and hence equal. Thus △TN'T' ∽△PNC.

$\angle NPH = \angle N'TP$ and $\angle NHP = \angle TPN'$ thus $\triangle NPH \backsim \triangle N'TP$. $\triangle PN'T'$ and $\triangle HNC$ have equal angles making them similar. Hence NH:HP :: N'P:TP and N'P:PT' :: NH:CH. From which
$NH^2{:}CH{\cdot}HP :: N'P^2{:}TP{\cdot}PT' = N'P^2{:}N'P{\cdot}CP = p/2{:}PP'/2 = p{:}PP'$.
This is the required relationship and proves that the method correctly located point N. We can now dispense with drawing the circle and any other part of the construction used for the proof that is not needed to find N. Consequently, to find N we need only to extend CP by p/2, locate i and duplicate the angle PiN' at O'.



This simplified method is illustrated in the figure to the left using a section with different PP' and PL to make the drawing more compact.

It shows how the angle PiN' can be copied by first swinging an arc of radius O'P from i intersecting iP at i* and iN' at i'. The arc length i*i' is then marked from P intersecting the semi-circle at N so that $\angle NO'P = \angle PiN'$.

Finally, NH is drawn through N parallel to PT. It doesn't get much simpler than this.

It seems reasonable to assume that Apollonius had many proven shortcut methods in his tool chest. In the remainder of proposition I.58 Apollonius constructs a mean proportional and a third proportional without comment. Using without comment a simple construction for NH, such as the one shown here, would thus not be unreasonable.

The reader can compare the method shown here with that proposed by Eutocius and draw their own conclusion as to Apollonius' probable procedure.

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